Berlin Lectures

Lecture 1. Hyperbolic Geometry

Let $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid y > 0 , \ z = x + iy \}$ upper half plane

The Riemannian metric
$$ds_h^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{y^2} \quad \text{on } \mathbb{H}^2$$

$v \in T_{(x,y)} \mathbb{H}^2$

$|v|^2_{\mathbb{H}^2} = |v|_E^2 / y^2$

- Angles in ds_h^2 = angles in $\mathbb{C}$. conformal
- length of $\gamma(t) = (x(t), y(t))$ $\quad t \in [a,b]$:

$$L(\gamma) = \int_a^b |\gamma'(t)|_{ds^2} \, dt$$

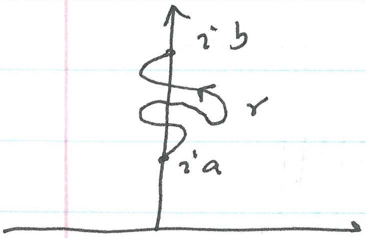
$$\gamma'(t) = (x'(t), y'(t))$$

$$|\gamma'(t)|_{ds^2} = \sqrt{x'(t)^2 + y'(t)^2} \Big/ y(t) \geq \frac{|y'(t)|}{y(t)}$$ equality iff $x' = 0$

so
$$L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} \, dt$$

Lemma 1. The positive y-axis is a geodesic in $\mathbb{H}^2 = (\mathbb{H}^2, ds^2)$ s.t
$$d(ia, ib) = \left| \ln \frac{b}{a} \right|$$

Pf Say $\gamma(a) = ia$, $\gamma(b) = ib$ $\quad a < b$ real $\quad , y(t) > 0,$



$$L(\gamma) \geq \int_a^b \frac{\sqrt{y'(t)^2}}{y(t)} \, dt = \int_a^b \frac{|y'(t)|}{y(t)} \, dt$$

$$\geq \left| \int_a^b \frac{y'(t)}{y(t)} \, dt \right| = \left| \ln y(t) \Big|_a^b \right| = \ln \frac{b}{a}$$

Equality holds iff $x'(t) \equiv 0$, $y'(t) \geq 0$ ie $\gamma([a,b]) \subset$ Y-axis + monotonic.

$\square$

Lemma 2. $f(z) = \frac{az+b}{cz+d}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$, $\mathbb{H} \to \mathbb{H}$ are isometries

Pf f is a composition of $z \mapsto z+b$, $z \mapsto \lambda z$ $\lambda \in \mathbb{R}$ and $z \mapsto -\frac{1}{z} = w$. Obviously $f(z) = \lambda z + b \in \mathrm{Iso}(\mathbb{H}^2)$. Now:

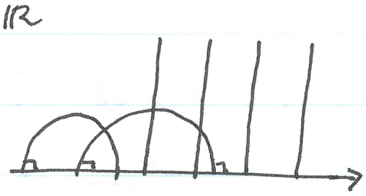
for $w = -\frac{1}{z}$ $dw = \frac{1}{z^2}\,dz \Rightarrow$ $\mathrm{Im}(w) = \frac{1}{2i}\left(\frac{1}{\bar z} - \frac{1}{z}\right)$

$$w^*(ds^2) = \frac{|dw|^2}{\mathrm{Im}(w)^2} = \frac{\frac{1}{|z|^4}\,|dz|^2}{\left(\frac{1}{2i}\right)^2\left(-\frac{1}{\bar z} + \frac{1}{z}\right)^2}$$

$$= \frac{\frac{1}{|z|^4}\,|dz|^2}{\frac{1}{|z|^4}\cdot\frac{1}{4}\,|\bar z - z|^2} = \frac{|dz|^2}{y^2}$$

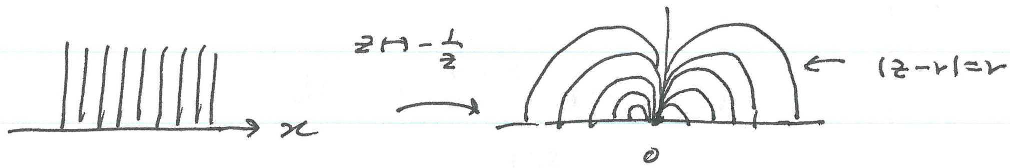
$\square$

Corollary 3. All geodesics in $\mathbb{H}$ are, either $\mathrm{Re}(z) = c$ or $|z-a| = r$, $a, c \in \mathbb{R}$



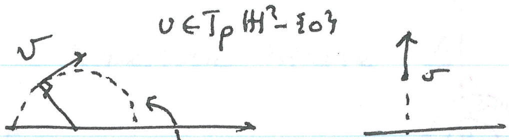
Pf: y-axis geodesic + $z \mapsto z+b$ iso $\Rightarrow$ $\mathrm{Re}(z) = c$ geod.

$\mathrm{Re}(z) = c$ geodesic + $z \mapsto -\frac{1}{z}$ iso $\Rightarrow$ $|z-r| = r$ geodesic



Now $z \mapsto z+b$ iso $\Rightarrow$ $\mathrm{Re}(z) = c$ & $|z-a| = r$ geodesic

These are all geodesics:



Geodesic determined by ONE tangent $\square$

<u>Homework</u>: Show that $PSL(2.\mathbb{R}) = SL(2.\mathbb{R})/\pm Id \cong Isd^+(\mathbb{H}^2)$

<u>hint</u>: $\forall$ $\gamma \in Iso^+(\mathbb{H}^2)$, find $f \in PSL(2.\mathbb{R})$ s.t

(1) $f(i) = \gamma(i)$

(2) $f'(i) = \gamma'(i)$.

<u>Cross ratio</u>: $a, b, c, d \in \hat{\mathbb{C}}$ distinct define

$$(a, b, c, d) = \frac{a-c}{a-d} : \frac{b-c}{b-d}$$

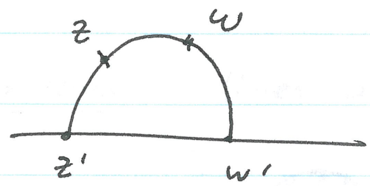
<u>We known</u> $f \in PSL(2.\mathbb{C})$, Möbius

$$f(z) = \frac{\alpha z+\beta}{\gamma z+\delta} \quad \begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix} \in SL(2.\mathbb{C}) \Rightarrow$$

$$(f(a), f(b), f(c), f(d)) = (a, b, c, d)$$

<u>Also</u> $\overset{a<b}{d(ia, ib)} = \ln\left(\frac{b}{a}\right) = \ln(ia, ib, \infty, 0)$ (1)

<u>Corollary 4</u>: If $z, w \in \mathbb{H}$, then $d(z.w) = \ln(z, w, w', z')$.



<u>Pf</u>: Let $f. \in PSL(2.\mathbb{R})$ sending $f(z) = ia$ $f(w) = ib$, then $f(z') = 0$ $f(w') = \infty$. <u>done</u>

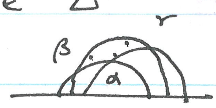
$$d(z.w) \overset{f\ iso}{=} d(ia, ib)$$

$$= \ln(ia, ib, \infty, 0)$$

$$\overset{f\ inv}{=} \ln(z, w, w', z')$$

$\square$

<u>Hw</u> Gauss-Bonnet: The area of a hyperbolic triangle Δ of angles α, β, γ is $\pi - \alpha - \beta - \gamma$



Hyperbolic Geometry $\mathbb{H}^2$

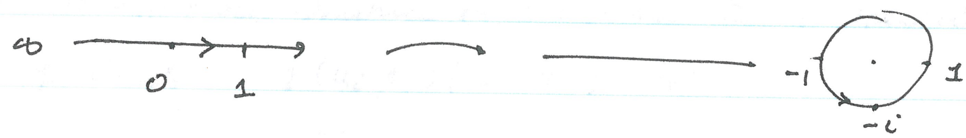
The ball model.

Note: $f(z) = (z, a, b, c)$ Möbius

$$= \frac{z-b}{z-c} : \frac{a-b}{a-c} \quad \text{sends} \quad \begin{array}{l} b \mapsto 0 \\ c \mapsto \infty \\ a \mapsto 1 \end{array}$$

So $f(z) = (z, 1, i, -1) = \frac{z-i}{z+1} : \frac{1-i}{2}$ $\quad 0 \mapsto$

Eg $\varphi(z) = \frac{z-i}{z+i}$: $\begin{array}{l} i \mapsto 0 \\ -i \mapsto \infty \\ 0 \mapsto -1 \\ \infty \mapsto 1 \end{array}$ and $1 \mapsto -i$

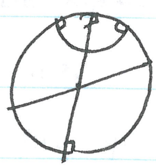


$\varphi : (\mathbb{H}) = \mathbb{D} = \{ |z| < 1 \}$

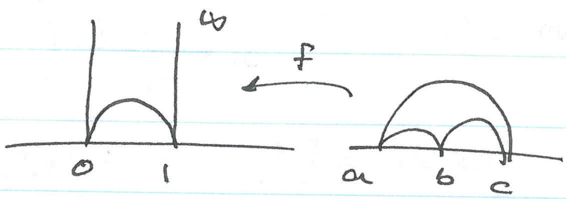
The metric on $\mathbb{D}$ making φ isometry: $\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = \frac{4|dz|^2}{(1-|z|^2)^2}$

Isometries: Möbius transf. preserving $\mathbb{D}$: $z \mapsto e^{i\theta}\left(\frac{z-a}{\bar{a}z-1}\right)$

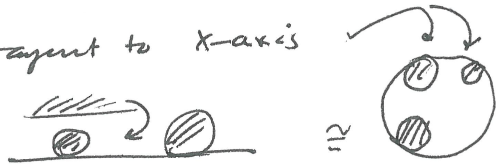
Geodesics: lines and circles $\perp \partial\mathbb{D}^2$



Eg Any two ideal triangles in $\mathbb{H}$ are isometric: all isometric to $(0, 1, \infty)$ due to $f(z) = (z, a, b, c)$. $a, b, c \in \mathbb{R}$ cross ratio (order)



<u>horoballs</u> Euclidean discs tangent to x-axis or $\mathrm{Im}(z) \geq c$



Lecture 2 Hyperbolic Structures on Surfaces

<u>Uniformization thm</u> Σ surface, connected, $\chi(\Sigma) < 0$, then $\forall$ Riemann metric g on Σ, $\exists !$ $u: \Sigma \to \mathbb{R}$ s.t. $(\Sigma, e^u g)$ is a complete hyperbolic metric. (Gaussian curvature $= -1$).

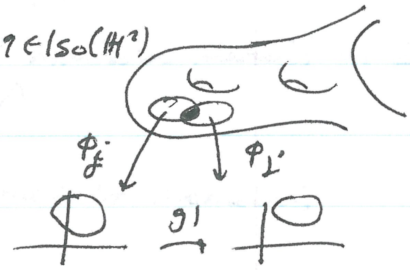
$\Rightarrow$ We should study hyperbolic metrics on surfaces & organize them.

<u>Def 1.</u> A hyperbolic structure on surface Σ : special collection of charts $\{(U_i, \phi_i) \mid i \in I\}$ s.t.

(1) $\Sigma = \bigcup_i U_i$

(2) $\phi_i : U_i \to \mathbb{H}^2$ 1-1 continuous

(3) $\phi_i \circ \phi_j^{-1} = g|_{\phi_j(U_i \cap U_j)}$ $g \in \mathrm{Isom}(\mathbb{H}^2)$



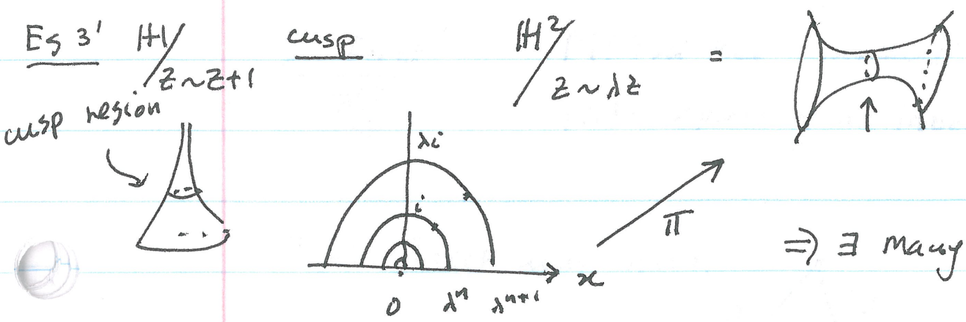
The structure is <u>complete</u> if each geodesic extends to ∞.

Geodesics, angles in (Σ, G) $\Leftarrow$ using <u>charts</u>

<u>Eg 2.</u> $(\mathbb{H}^2, ds^2)$ complete hyperbolic $d(ia, it) \to +\infty$ $t \to +\infty$ or <u>$t \to 0^+$</u>

<u>Eg 3.</u> $\gamma(z) = \lambda z$, $\lambda > 1$ acts on $\mathbb{H}^2$ generates a group $\mathbb{Z} = \langle \gamma^n \rangle$. the quotient space

<u>Eg 3'</u> $\mathbb{H}/_{z \sim z+1}$ <u>cusp</u> $\qquad \mathbb{H}^2/_{z \sim \lambda z} =$ <u>a hyperbolic annulus</u>



length of the shortest geod $\boxed{\ln \lambda}$

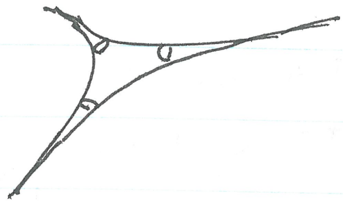
non-isometric

$\Rightarrow \exists$ many distinct hyperbolic structures on $S^1 \times \mathbb{R}$.

<u>Eg 4.</u> $\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}) \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod 2 \right\} / \pm Id$

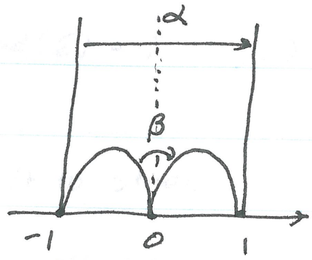
L2

$\Gamma(2)$ acts on $\mathbb{H}^2$ freely properly discontinuously w/ quotient

$\mathbb{H}^2/\Gamma(2) \cong$  metric double of an ideal triangle $\cong \mathbb{C} - \{0,1\}$

Sketch:

$\Gamma(2)$ generated by $\alpha(z) = z+2 \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and
$\parallel$
$\mathbb{Z} * \mathbb{Z}$ $\qquad \beta(z) = \dfrac{z}{2z+1}.$ $\qquad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

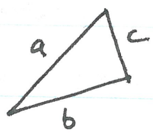


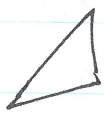
$\beta:\ 0 \mapsto 0$
$\quad -1 \mapsto 1$

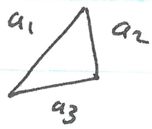
$\Rightarrow$ Schottky group $\Rightarrow$ result.

What is Teichmuller theory?: space of all ^equivalence classes hyperbolic metrics on Σ.

Eg 5. Let $M(\Delta)$ = space of all triangles in $\mathbb{E}^2$ modulo isometries





use 

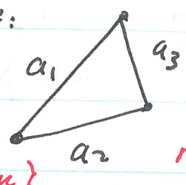
Q Is $M(\Delta) = \{(a,b,c) \in \mathbb{R}^3_{>0} \mid a+b>c,\ b+c>a,\ c+a>b\}$?

ANS: Is No: $M(\Delta) \neq \{(a_1, a_2, a_3) \in \mathbb{R}^3_{>0} \mid a_i + a_j > a_k\}$.

In fact $M(\Delta)$ is the "moduli" space (Riemann)

$T(\Delta)$ = space of all labelled trigs in $\mathbb{E}^2$ modulo isometries preserving labelling

$T(\Delta)$ = Teichmuller space.



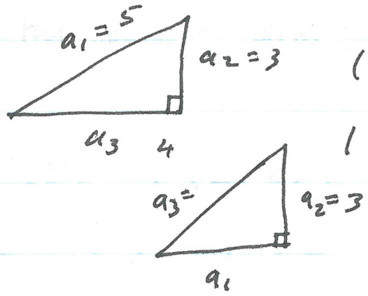
We know first second third edges $v_1\ v_2\ v_3$)

what
so space $\{(a_1 a_2 a_3) \in \mathbb{R}^3_{>0} \mid a_i + a_j > a_k\} \longleftrightarrow$ marked trigs (verts

Modulo isometry preserving marking

$M = T/S_3$ S_3 the permutation group

ie



$(5, 4, 3) \in T$ &

$(4, 3, 5) \in T$ Are different

But they are the same in M: isometric

The ultimate goal: (Riemann's moduli space)

$$\mathrm{Mod}(\Sigma) = \{ (\Sigma, d) \mid d \text{ complete } \underbrace{\text{hyperbolic}}_{\text{finite area}} \text{ metric on } \Sigma \} / \text{isometry}$$

Very difficult to study.

Teichmuller space

$$T(\Sigma) = \{ (\Sigma, d) \mid (\Sigma, d) \underset{T}{\cong} (\Sigma, d') \text{ if } \exists \text{ isometry } h \simeq id \}$$

$$= \{ (\Sigma, d) \mid d - \} / \text{isometry homotopic to id.}$$

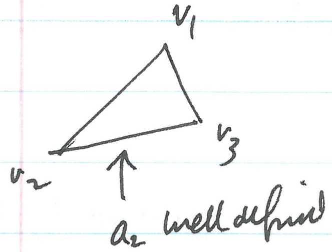
Why $T(\Sigma)$? Just like $T(\Delta)$: we can define the length of a loop (edge). The length of the 2nd edge function does not make sense in $M(\Delta)$!

So $\forall$ loop $\alpha \subset \Sigma$

$$l_\alpha : T(\Sigma) \longrightarrow \mathbb{R} \quad (\Sigma, d) \longmapsto \text{length of the shortest path } \alpha' \simeq \alpha \text{ in } d$$

Why Teich(Δ)? We can talk about the i-th edge lengths

$a_i : \mathrm{Teich}(\Delta) \longrightarrow \mathbb{R}_{>0}$



No such function on $\mathrm{Mod}(\Delta)$? $\longrightarrow \min\{a_1, a_2, a_3\}$

Only the minimal length. $\uparrow$ C^0

Not Smooth

Lecture 2 Berlin 9-13-2013

Σ orientable

(1)

(1)

$d =$ complete finite area hyperbolic

Main goal Σ topological surface ~~w/ complete hyperbolic structure~~ ~~finite~~ area

$$\mathrm{mod}(\Sigma) = \{ (\Sigma, d) \mid d \text{ complete hyperbolic finite area} \} / \mathrm{iso}$$

What is the space ? dim ? connected ? Topology ? Geometry ?

$$T(\Sigma) = \{ (\Sigma, d) \mid d. \text{ ——} \} / \text{isometry homotopic to id}$$

i.e. $(\Sigma, d) \underset{\text{Teich}}{\sim} (\Sigma, d')$ if $\exists$ iso $h : (\Sigma, d) \to (\Sigma, d')$

s.t. $h \simeq id$

So $\mathrm{mod}(\Sigma) = T(\Sigma) / MCG(\Sigma)$

$MCG = \{$ orientation preserving homeo $h \} / \{ h \mid h \simeq id \}$

This plays the role of change of marking S_3 for Δ.

Key Fact: Each loop $\alpha \subset \Sigma$ $\exists$ lengths $\ell_\alpha : T(\Sigma) \to \mathbb{R}$ $\quad \beta \simeq \alpha$

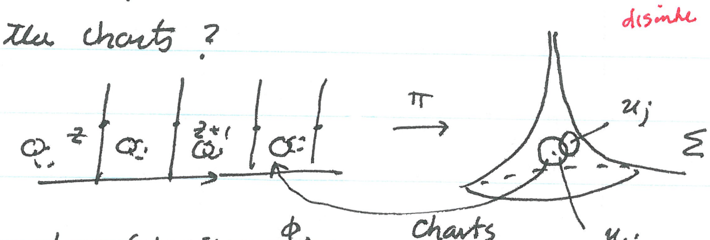
$[d] \to$ length of the shortest

(2) Many one words about hyperbolic st. on $\mathbb{H}/\Gamma$ $\quad \Gamma < \mathrm{Isom}^+(\mathbb{H})$

discrete

How to see the charts ?

Eg. $\Sigma = \mathbb{H} / z \sim z+1$



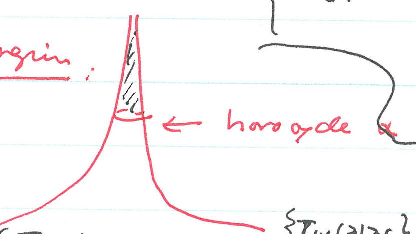
$\phi_i = (\pi|_{V_i})^{-1}$, $\phi_j = (\pi|_{V_j})^{-1}$

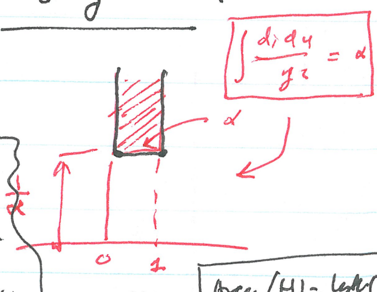
$\phi_i \circ \phi_j^{-1} \in \langle \gamma \rangle$!

More generally: $\mathbb{H}/\Gamma$ has charts $\{ (u_i, \phi_i) \mid i \in I \}$

st $\phi_i \circ \phi_j^{-1} \in \Gamma$!

Finite area of cusp region:





~~A cusp~~ A horoball in (Σ, d)

$\cong$ isometric to

Area$(H) =$ length(∂H)

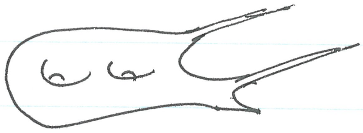
S_g ~~[illegible]~~ closed surface $V=\{v_1,\dots,v_n\}\subset S$ $\Sigma = S_g - V \cong \Sigma_{g,n}$

~~[illegible]~~ oriented $\chi(\Sigma)<0$

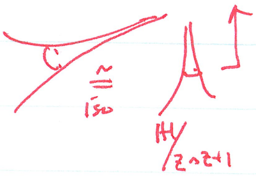
L3 Topological Triangulations

Q. How to construct all complete finite area hyperbolic structures on Σ which is non-closed? w/ $\chi(\Sigma)<0$. ? $\boxed{\text{Known}}$ finite area $\Rightarrow$ cusp end!

$\Sigma = \Sigma_g - \{v_1,\dots,v_n\}$ $n\geq 1$ $\chi(\Sigma)<0 \Leftrightarrow$ $g\geq 1$ or $g=0$ $n\geq 3$.

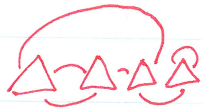


$V=\{v_1,\dots,v_n\}$



ANS Use triangulations

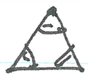
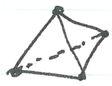
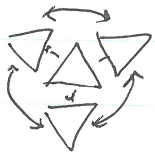
2-Dimensional triangulations.



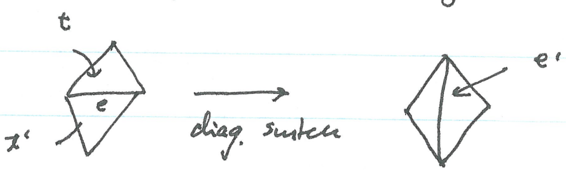
Take a finite collection of disjoint oriented triangles $\Delta_1,\dots,\Delta_k$, identify pairs of edges in $\sqcup\Delta_i$ by orientation reversing homeomorphisms, The quotient space $S=\sqcup\Delta_i/\sim$ is a compact oriented surface w/ a triangulation $\mathcal{T}$: simplexes in $\mathcal{T}$ $\Leftrightarrow$ quotient of simplexes in $\sqcup\Delta_i$.

Let V, E be the sets of vertices and edges respectively in $\mathcal{T}$.
$V = V(\mathcal{T})$, $E = E(\mathcal{T})$

We say $\mathcal{T}$ an ideal triangulation of $S-V$.

Eg  double of a triangle, or  ∂(tetra). 

Def Diagonal switch: If $e\in E$ is adjacent to two triangles t, t', then



$\mathcal{T}\to\mathcal{T}'$ by replacing e by the other diagonal e' in $t\cup t'$.

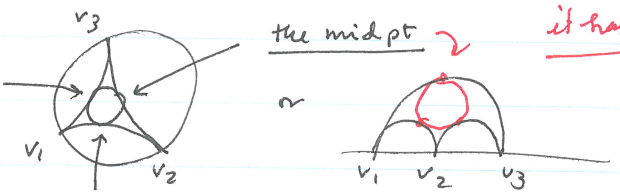
Thm If $\mathcal{T},\mathcal{T}'$ two triangulations of S with the same set of vertices $V(\mathcal{T})=V(\mathcal{T}')$, then $\mathcal{T}$ and $\mathcal{T}'$ are related by a sequence of diagonal switches.

Each $\Sigma = \Sigma_{g,n}$ $2-2g-n<0$ has an ideal triangulation $(n \geq 0)$

Q Suppose $(\Sigma, \mathcal{T})$ ideal triangulated surface, How to use $\mathcal{T}$ to produce all hyperbolic metrics on Σ ?

Thurston's work

Ideal triangle: Convex hull of 3 points $v_1, v_2, v_3 \in \partial \mathbb{H}^2$



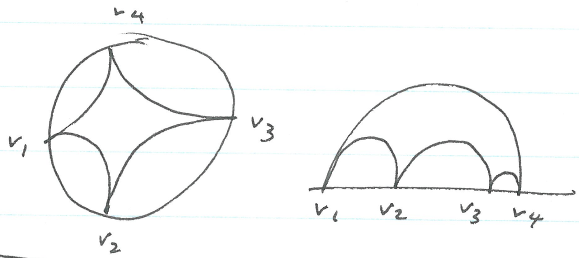
it has 3 mid pts

ANY two of them are isometric

$f(z) = (z, v_1, v_2, v_3)$

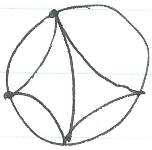
drop

Ideal quadrilateral: Convex hull of 4 points $v_1, v_2, v_3, v_4 \in \partial \mathbb{H}^2$



Not all are isometric since the cross ratio (v_1, v_2, v_3, v_4) is an isometry invariant

A marked ideal quad δ ~~an ideal quad & diagonal~~ e

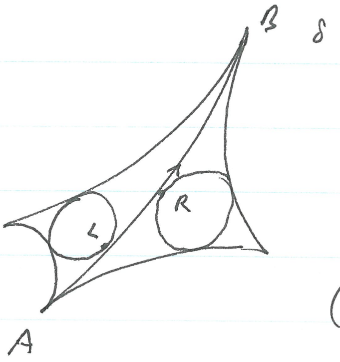


= union of two ideal triangles along an edge e.

Every surface is oriented

Def. Thurston's shear coordinate $d(\delta)$ of an oriented marked ideal quad δ is the real number $d(\delta) =$ signed distance from L to R along the diagonal.

L3. Thurston's Shear Coordinate

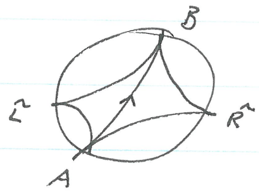


orient e : $A \to B$: $L \to R$

so $d(\delta) = R - L$ computed along e

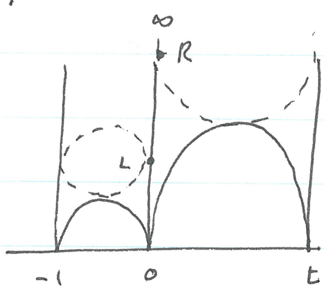
Note independent of the choice of orientation on e (depending only on the orientation of δ!)

Lemma 1. Suppose $\delta = [A, \hat{R}, B, \hat{L}]$ as shown



then
$$d(\delta) = \ln[-(A, B, \hat{R}, \hat{L})]$$

Proof May assume after a Möbius transf $A = 0$, $B = \infty$, $\hat{R} = t > 0$, $\hat{L} = -1$



$$(A, B, \hat{R}, \hat{L}) = (0, \infty, t, -1)$$
$$= \frac{0-t}{0+1} : \frac{\infty - t}{\infty - 1} = -t$$

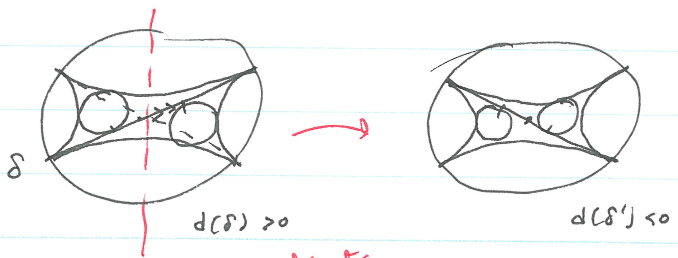
$L = i$ $\quad R = it$

so $d(\delta) = \ln(t)$

$\square$

Lemma 2. Suppose δ' is obtained from δ by a diagonal switch $\Rightarrow$ $d(\delta') = -d(\delta)$

Pf Follows from the basic property of cross ratio. Or a geometric proof:



$|d(\delta)| = |d(\delta')|$.

hyperbolic reflection
$(x, y) \mapsto (-x, y)$
iso

$\square$

L4 The Shear coordinate of Thurston

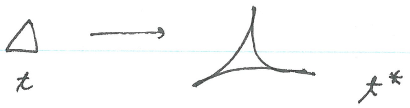
Assume $\Sigma = \Sigma_g - \{v_1, \dots, v_n\}$ $n \geq 1$, $\chi(\Sigma) < 0$

$(\Sigma, \mathcal{T})$ ideal triangulation

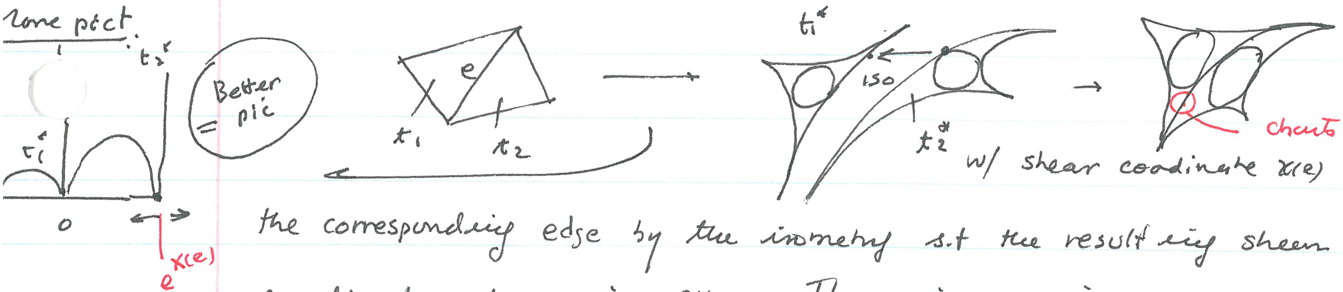
$\boxed{\text{Fixed}}$ $\mathcal{T}$

For each $x \in \mathbb{R}^{E(\mathcal{T})}$ i.e. $x : E(\mathcal{T}) \to \mathbb{R}$, one produce a (possibly incomplete) hyperbolic structure $\pi(x)$ on Σ as follows

(1) replace each triangle $\Delta \in \mathcal{T}$ by an ideal hyperbolic triangle

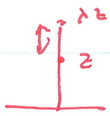


(2) each edge $e \in E(\mathcal{T})$ with $x(e)$ assigned, glue isometrically t_1^* t_2^*



the corresponding edge by the isometry s.t the resulting shear coordinate at e is $x(e)$. There is a unique way to glue.

$\boxed{\text{infinite geodesic}}$ $L = \{ \mathrm{Re}(z) = 0 \}$

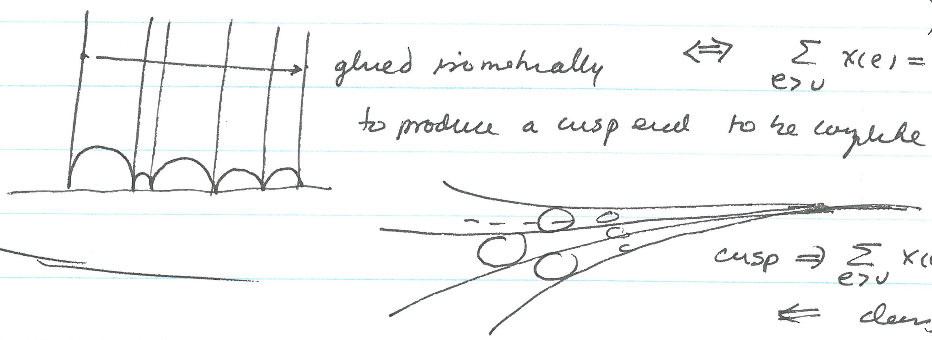


$\boxed{\ln \lambda}$ $f_\lambda(z) = \lambda z : L \to L$ translation $\boxed{d(z, f_\lambda(z)) = \ln \lambda}$

finite area

<u>Lemma</u> $\pi(x)$ is a complete metric $\Leftrightarrow$ $\forall\, v \in V = \{v_1, \dots, v_n\}$
$$\sum_{e > v} x(e) = 0$$
$\Updownarrow$ <u>cusp end</u>

<u>Proof</u> At v_i say put it at ∞



glued isometrically $\Leftrightarrow$ $\sum_{e > v} x(e) = 0$

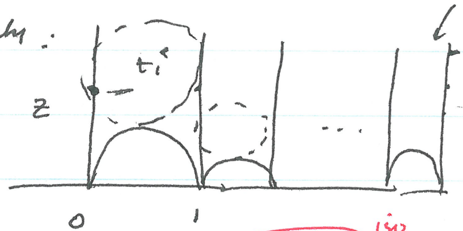
to produce a cusp end to be complete

cusp $\Rightarrow \sum_{e > v} x(e) = 0$

$\Leftarrow$ clear.

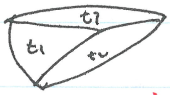
Pf Sketch. Topology  at ∞

Geometry:

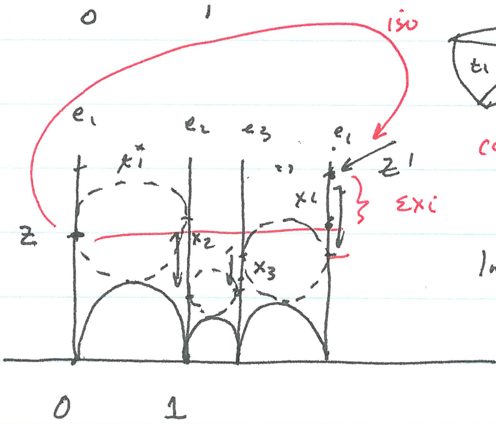


idealized

z' & heights

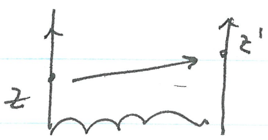


corresponding pts



$\underline{z \sim z'}$ by the construction

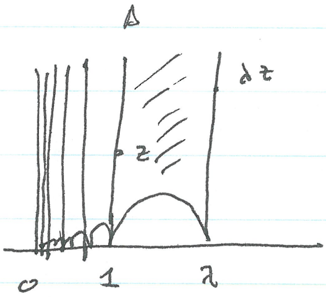
$\mathrm{Im}\, z' / \mathrm{Im}\, z = e^{x_1+\dots+x_3}$



must be horocycle in order to produce cusp

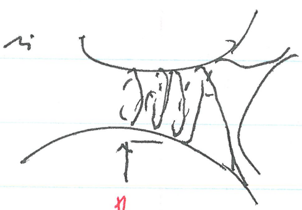
$\Rightarrow \boxed{\sum x_i = 0}$

Eg: The image of the lines



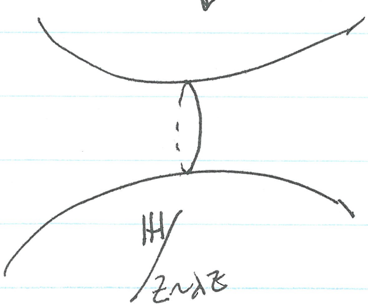
$\lambda > 1$

$\sim \mathbb{H}/z \sim \lambda z \sim$

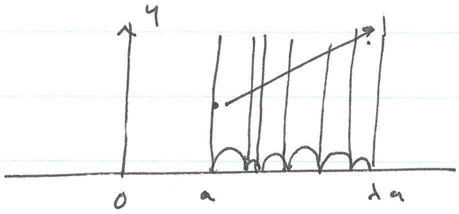


incomplete structure

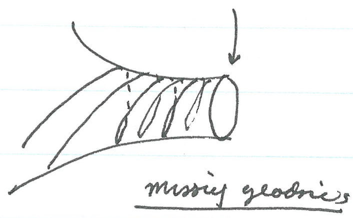
$\downarrow \pi$ quasi-image of A spiral



Note if $\sum_{e>v} x(e) \neq 0$, the gluing is



⇒ the infinity



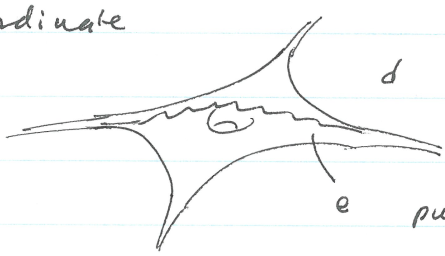
incomplete structure

∎

Let $\mathbb{R}^E_p = \{ x \in \mathbb{R}^E \mid \forall v \in V \quad \sum_{e>v} x(e) = 0 \}$ (linear subspace)

<u>Thm</u> (Thurston) The map $\Phi_{\mathcal{T}} : \mathbb{R}^E_p \longrightarrow T(\Sigma)$ $\quad x \mapsto [\pi(x)]$
is a ~~homeomorphism~~ 1-1 onto map. $\Phi(x)$ has <u>shear coordinate x</u> w.r.t. $\mathcal{T}$.

<u>Proof</u> Φ is onto: If $[d] \in T(\Sigma)$ + $\mathcal{T}$ ideal triangulation $\mathcal{T}$
We can isotopy $\mathcal{T}$ to be a geodesic triangulation of (Σ, d)
where all edges are infinite simple geodesics ⇒ x = shear coordinate
from cusp to cusp

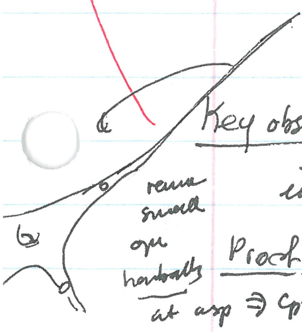


pull tight ⇒ <u>done</u>

" <u>Φ is 1-1</u>": if $\phi(x) = \phi(x')$ ⇒ they have the same shear coordinate. $\boxed{\text{How to see it?}}$

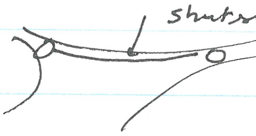
For (Σ, d), in (Σ, d)

<u>Key observation</u> Each edge $e \in \mathcal{E}(\mathcal{T})$ is homotopic to a unique infinite geodesic e^*.



<u>Proof</u> Pull tight by producing horocycles ∎

remove small open horoballs at cusp ⇒ cpt X



$\boxed{h(e) \simeq e}$

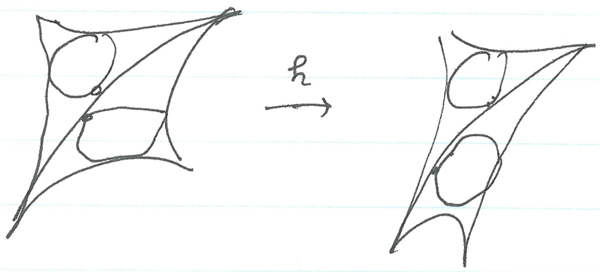
$\underline{\Phi \text{ is } 1\text{-}1}$: $\exists$ an isometry

$h : (\Sigma, \Phi(x)) \to (\Sigma, \Phi(x'))$

$h \simeq id$ ⇓ Φ circulate

$\Rightarrow$ $\underline{h(\mathcal{T}_x) = \mathcal{T}_{x'}}$ $\quad$ ($\because$ $\mathcal{T}_x$, $\mathcal{T}_{x'}$ geodesic triangulations in $\Phi(x)$ & $\Phi(x')$ metrics)

property



must preserve the distance

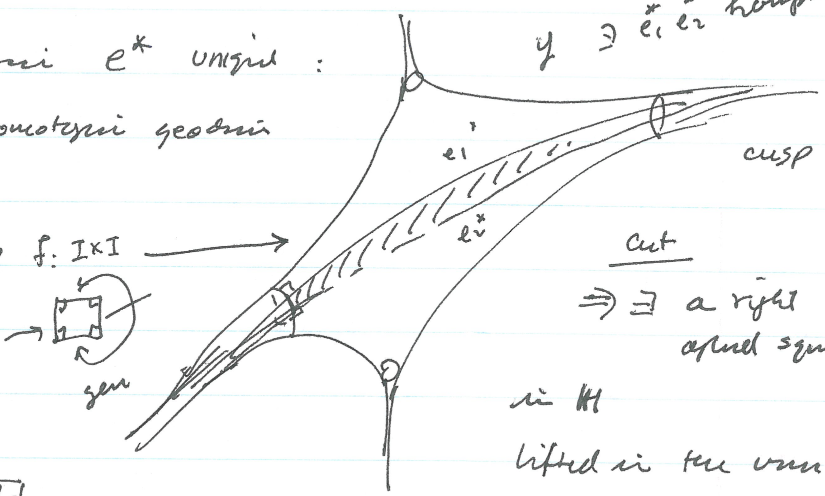
The point is that $\mathcal{T}_x$ is unique!

Why is the geodesic e^* unique :

Two homotopic geodesics

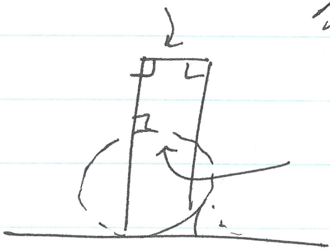
If $\exists$ e_1^* e_2^* homotopic

$\exists$ a map $f : I \times I \to$



cut

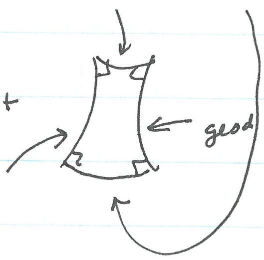
$\Rightarrow$ $\exists$ a right angled square in $\mathbb{H}$ lifted in the universal cover



horocycles $\underline{\text{impossible}}$

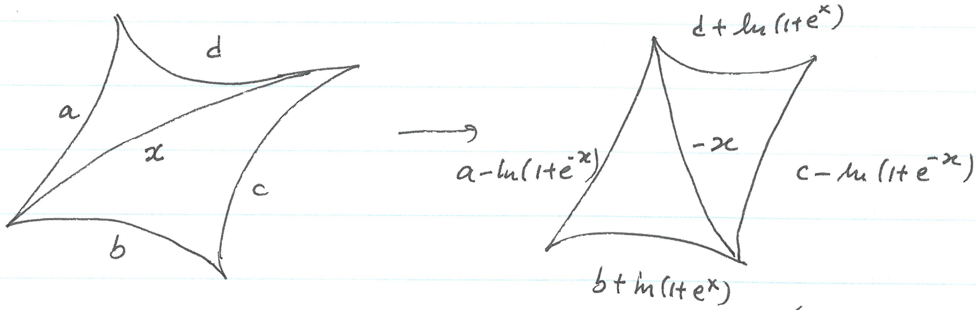
universal cover

$\Rightarrow$ $\exists$ a right-angled quadrilateral $Q \subset \mathbb{H}$ s.t



skip it for Now

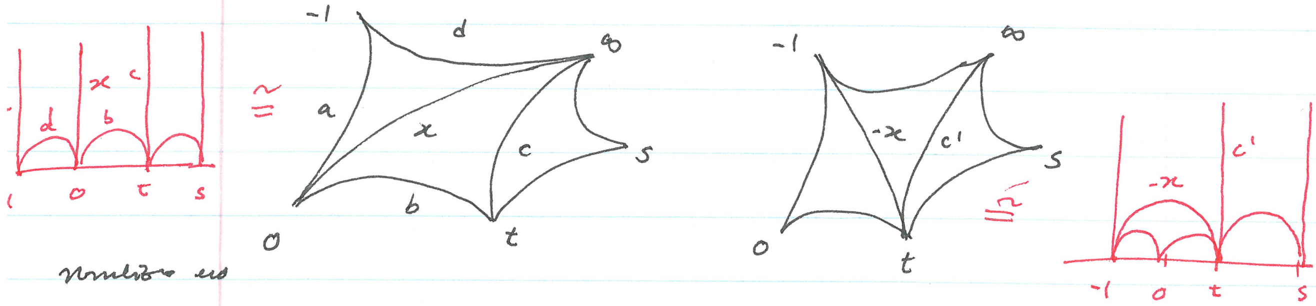
Thm If $\mathcal{T}, \mathcal{T}'$ two triangulations related by a diagonal switch, then the transition function $\Phi_{\mathcal{T}'}^{-1}\,\Phi_{\mathcal{T}} : \mathbb{R}_p^{E(\mathcal{T})} \to \mathbb{R}_p^{E(\mathcal{T}')}$ takes the following form: real analytic diffeo



all others places remain the same. (Right-hand orientation)

Proof Follows from the basic cross ratio rule. $\square$

We may assume the end points are: $\underline{s > t > 0}$



Here: $x = \ln t$

$$c = \ln(-1)(t, \infty, s, 0) = \ln\left(\frac{t-s}{t-0}\right)(-1) = \ln\left(\frac{s-t}{t}\right)$$

$$c' = \ln\Big[(-1)(t, \infty, s, -1)\Big] = \ln(-1)\,\frac{t-s}{t+1} = \ln\frac{s-t}{(t+1)}$$

$$= \ln\left(\frac{s-t}{t}\right) + \ln\left(\frac{t}{1+t}\right) = c - \ln\left(1+\frac{1}{t}\right) = c - \ln(1+e^{-x})$$

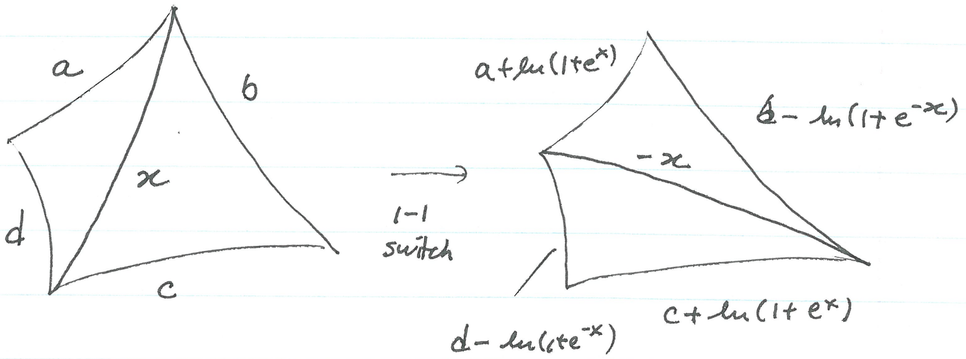
$\square$

RM: Basic rule: Right "shoulder" Add $\ln(1+e^{x})$, left: $-\ln(1+e^{-x})$

Corollary $\Rightarrow$ $\mathcal{T}(S)$ is a real analytic mfd diffeo $\mathbb{R}^{6g-6+2n}$. $= x - \ln(1+e^{x})$

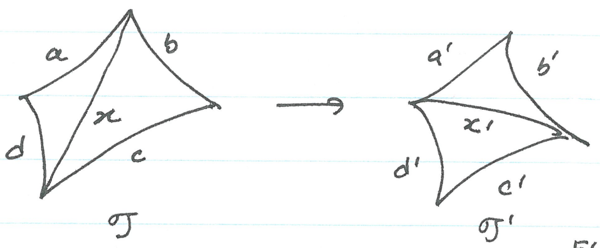
L5. Penner's λ-length Coordinates, Poisson Structures. Conjecture.

Summary the change of coordinate formula for Thurston shear coord.



Any property invariant under the coordinate change is an invariant of the Teichmüller space.

(HW) <u>Thm</u>. Suppose $f(t)$, $g(t)$ smooth s.t. $f(t)+g(t) = t$ and a coordinate change rule:



with $x' = -x$
$a' = a + f(x)$
$c' = c + f(x)$
$b' = b + g(x)$
$d' = d + g(x)$

Then the Poisson structure on $\mathbb{R}^{E(\sigma)}$

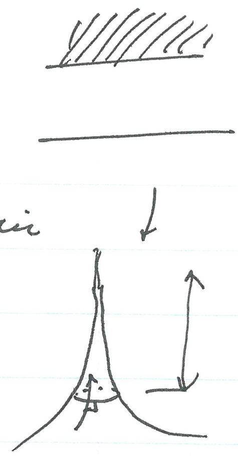
$$\sum_{e \to e'} \frac{\partial}{\partial x_e} \wedge \frac{\partial}{\partial x_{e'}}$$

is invariant under coordinate change (the diagonal switch)

where $e \to e'$ means $\exists\ \Delta \in \sigma^{(2)}$ adjacent to both e + e' + $e \to e'$ in the orientation of Δ.

Homework
Just do it

($\Rightarrow$ $T(\Sigma)$ is naturally a symplectic mfd)

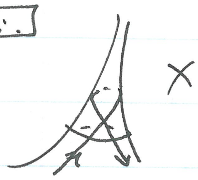


Recall $\qquad d$ — complete finite area hyperbolic

A horoball $H \subset (\Sigma, d)$ is a subsurface isometric to $\{Im(z) \geq c\}/(z \sim z+1)$

Fact: (Σ, d) non cpt. ~~nbhd of infinity of near each~~ v_i.

horoballs form nbhds of infinity of Σ



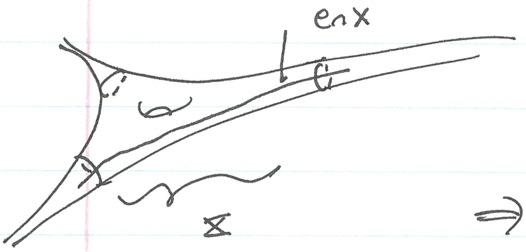
- $Area(H) = length(\partial H)$ (Homework)

- (Σ, d) take $H_1, \ldots, H_n$ disjoint horoball nbhd of v_i.

$X = \Sigma - \bigcup_i H_i$ is a compact surface w/ ∂H horocycles.

Key lemma (Σ, d). $\mathcal{T}$ ideal triangulation $\forall e \in E(\mathcal{T})$ is homotopic to a unique geodesic e^* in d.

Pf existence $\quad$ Form X + pull tight $e \cap X$



Let α be path in X

st ① $\alpha \simeq e \cap X$ rel(∂X)

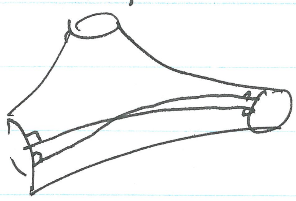
② α has the shortest length

$\Rightarrow \alpha$ geod. + $\alpha \perp \partial X$ (X cpt)

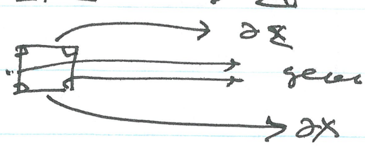
$\Rightarrow \alpha$ extends to a geodesic e^*

going to the cusp

Uniqueness $\quad$ if $\exists\ e_1^* \simeq e_2^*$ homotopic geodesics both go into cusps v_i to v_j $\Rightarrow e_1^* \cap X \simeq e_2^* \cap X$ rel(∂X)

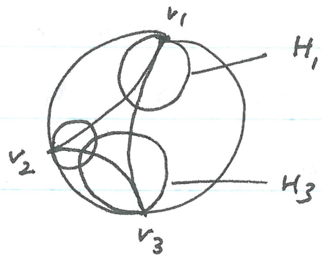


$\Rightarrow \exists\ f: I \times I \longrightarrow \Sigma$ $\quad$ lift it to $\mathbb{H}^2$

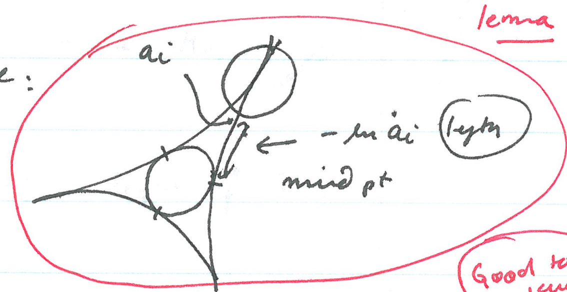


L5 Penner's λ-coordinate

A decorated ideal triangle τ: ideal triangle w/ vertices $v_1, v_2, v_3 \in \partial\mathbb{H}^2$ s.t each v_i is associated with a horoball H_i at v_i.

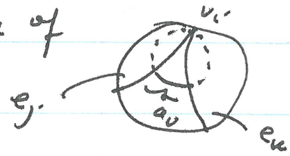


picture:

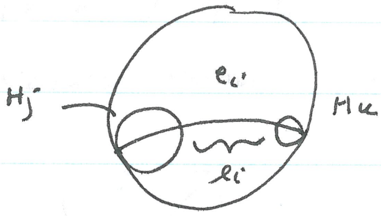


Let the edges of τ be e_1, e_2, e_3 (infinite geodesics) e_i opposite to v_i.

Then the angle a_i of τ at v_i is: the length of

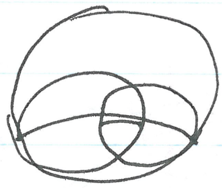


The length l_i of τ at e_i $\quad l_i \in \mathbb{R}$



l_i = length ≥ 0

$H_j \cap H_k = \phi$



$-l_i$ = length.

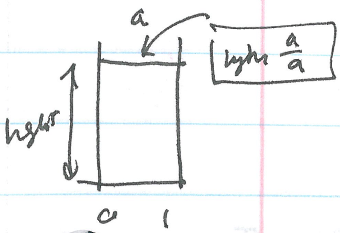
Penner $L_i = e^{\frac{1}{2}l_i} \in \mathbb{R}_{>0}$ the λ-length of e_i

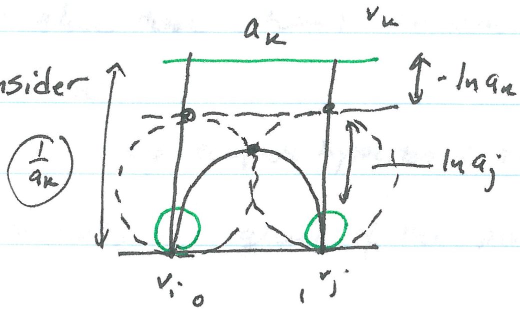
Lemma (1) $a_i = e^{\frac{1}{2}(l_i - l_j - l_k)} = \dfrac{L_i}{L_j L_k}$ $\qquad \ln a_i = \frac{1}{2}\left(\ln l_i - \ln l_j - \ln l_k\right)$

(2) $-\ln a_i$ = dist H_i to mid pt

(3) $\forall\, l_1, l_2, l_3 \in \mathbb{R}$, $\exists!$ decorated ideal tetra of lengths l_1, l_2, l_3 (HW)

Proof Consider





$\Rightarrow$ $\boxed{+l_i = -(\ln a_j + \ln a_k)}$

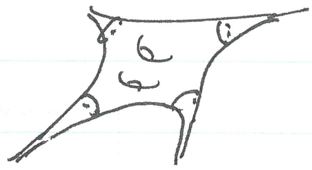
$\Rightarrow$ result.

(2) The angles $a_i, a_j, a_k \in \mathbb{R}_{>0}$ can be arbitrary real numbers as shown above $\Rightarrow$ result.

□

λ-length

A decorated hyperbolic metric (d, w) on Σ:



d — complete finite area hyperbolic on Σ.

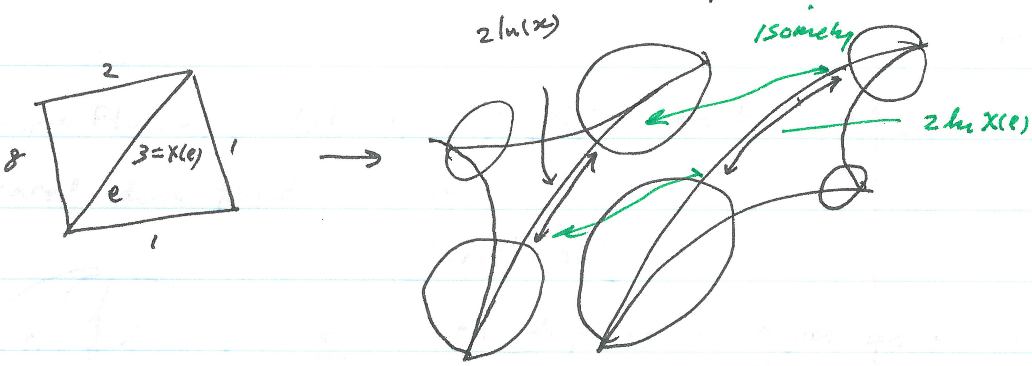
w: each cusp is assigned a horoball H_v centered at the cusp.

$w = (w_1, \cdots, w_n) \in \mathbb{R}^n_{>0}$. w_i = length of ∂H_i

$$T_D = \{[(d,w)] \mid (d,w) \text{ decorated metric on } \Sigma\} / \text{isometry} \simeq \text{id preserving horoballs}$$

$$= T(\Sigma) \times \mathbb{R}^n_{>0}$$

Now fix a $\mathcal{T}$ of Σ. For any $x \in \mathbb{R}^{E(\mathcal{T})}_{>0}$ one construct a decorated metric $\varphi(x) \in T_D$ as follows.



Make each $\Delta \in \mathcal{T}^{(2)}$ a decorated ideal tetra of λ-length $x(e)$.

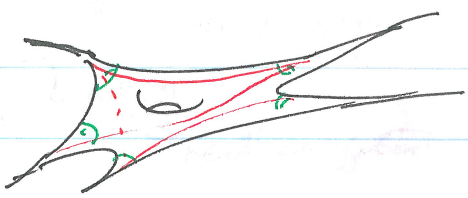
Now glue these (them) isometrically along edges preserving decorations

$\Rightarrow$ complete hyperbolic metric of finite area (d, w)

The horoballs — gluing of the portion of the horoballs.

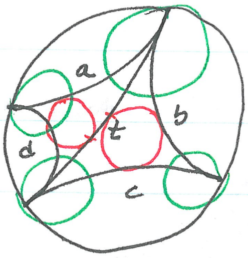
Thm (Penner) $\Phi_{\mathcal{T}}: \mathbb{R}^E \longrightarrow T_D(\Sigma) : x \mapsto \varphi(x)$ is a homeomorphism.

Proof Onto: pull $\mathcal{T}$ tight. 1-1 definition.



Relationship between shear and λ- coordinate:

Lemma: Consider $a, b, c, d, t \in \mathbb{R}$ the length coord.

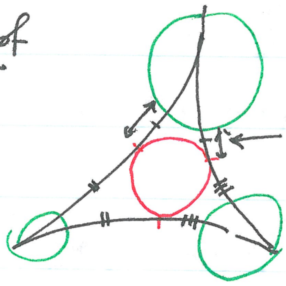


then the shear coordinate x at e

$$x = \cancel{(a+c+b+d)/2} \left(\frac{b+d-a-c}{2}\right)$$

From Penner → shear

Proof



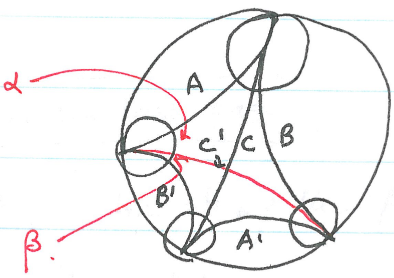
$\dfrac{-c+b+t}{2}$ by definition.

so $x = \left(\dfrac{-c+b+t}{2}\right) - \left(\dfrac{-d+a+t}{2}\right) = \dfrac{1}{2}(b+d-a-c)$

□

Corollary (Penner's Ptolemy). Let A, A', B, B', C, C' be the λ-lengths of decorated ideal quad. Then

$$CC' = AA' + BB'$$



Proof Let α, β be the angles as shown

By the cosine law:

$$\alpha = \frac{B}{AC'} \qquad \beta = \frac{A'}{B'C'}$$

But $\alpha + \beta = \dfrac{c}{AB'}$ ⇒

$$\frac{B}{AC'} + \frac{A'}{B'C'} = \frac{c}{AB'} \iff AA' + BB' = CC'$$

$\hat{\Phi}_T \circ \Phi(\frac{\varphi}{2})$ □

⇒ the change of coordinate formula $\tilde{\Phi}_{\sigma} \circ \tilde{\Phi}_{\sigma'}^{-1}$: even better the λ-lengths